Measurement of the Probability Distribution of Total Transmission in Random Waveguides

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Measurements have been made of the probability distribution of total transmission of microwave radiation in waveguides filled with randomly positioned scatterers which would have values of the dimensionless conductance g near unity. The distributions are markedly non-Gaussian and have exponential tails. The measured distributions are accurately described by diagrammatic and random matrix calculations carried out for nonabsorbing samples in the limit $g \gg 1$ when g is expressed in terms of the variance of the distribution, which equals the degree of long-range intensity correlation across the output face of the sample.

42.25.Bs, 42.68.Mj, 41.20.Jb

Nonlocal correlation in the flux transmitted through mesoscopic samples leads to enhanced fluctuations of local and spatially averaged transmission for both classical and quantum waves. [1,2] Such fluctuations increase dramatically as the ensemble average of the dimensionless conductance, g, approaches unity. Low values of g can be achieved in quasi-one-dimensional samples such as conducting wires or multimode waveguides with lengths much greater than the transverse dimensions. In this Letter, we report measurements of the probability distribution of total transmission of microwave radiation in long waveguides filled with randomly positioned scatterers which in the absence of absorption would have values of g near unity. The distributions observed are markedly non-Gaussian. They are compared to recent diagrammatic and random-matrix calculations for nonabsorbing samples in the limit $g \gg 1$ [3,4]. This is done by reexpressing the distribution, which is a function of the single parameter g, as a function of the variance of the normalized transmission using the relation between these parameters. This result is in excellent agreement with the measured transmission distributions and indicates that the variance of the normalized transmission, which equals the degree of long-range intensity correlation across the output face of the sample, is the essential parameter describing fluctuations in random media.

Key transmission quantities in order of increasing spatial averaging are the intensity, T_{ab}, which is the transmission coefficient for incoming mode a into mode b, the total transmission for incoming mode a, $T_a = \sum_b T_{ab} \sim \ell/L$, and the total transmittance $T = \sum_{ab} T_{ab} \sim N\ell/L$, where ℓ is the transport mean free path, L is the sample length, and N is the number of modes. The total transmittance is equivalent to the dimensionless conductance in electronic systems, $T = G/(e^2/h)$, where G is the conductance [5] and $g = <T> = N\ell/L$. Though the variances of the transmission quantities normalized to their ensemble average values, $s_{ab} = T_{ab}/<T_{ab}>$, $s_a = T_a/<T_a>$ and $s = T/<T>$, are reduced as the extent of spatial averaging increases, fluctuations in these quantities do not self average, as they would if spatial correlation were absent. To leading order in $1/g$, the enhancement of the variances of s_{ab}, s_a, and s arising from nonlocal correlation is 1, L/ℓ [6], and $(L/\ell)^2$ [7,8], respectively, which results in values of the variances of 1, $1/g$, and $(1/g)^2$. [9,10]

To examine the scaling and the universality of transport, it is important to measure the full distribution of key transmission quantities as the sample size, and hence g, changes. In previous work, nonlocal correlation has been shown to lead to higher probabilities at large values of the intensity, leading to a deviation from negative exponential statistics for polarized microwave radiation when $g \sim 10$ [11,12], as well as to discernable deviations from a Gaussian distribution and enhanced variance for the total optical transmission when $g > 10^3$ [13].

Recently, an expression for $P(s_a)$ in terms of g for nonabsorbing samples was obtained by Nieuwenhuizen and van Rossum using diagrammatic techniques combined with random matrix theory [3] and subsequently by Kogan and Kaveh within the framework of random matrix theory [4]. The diagrammatic calculations neglect some terms of order higher than $1/g$, whereas computations based on random matrix theory neglect sample-to-sample fluctuations in the probability distributions of eigenvalues of the transmission matrix and are expected to be accurate only to order $1/g$. More recently, van Langen, Brouwer and Beenakker carried out a nonperturbative calculation of the total transmission distribution in the absence of absorption [14]. An analytic solution is obtained for the case in which time reversal invariance is broken ($\beta = 2$) but not for

the case of time reversal symmetry ($\beta = 1$) considered here. However, good agreement is found between the β-independent result for $P(s_a)$ obtained in Refs. [3,4] and the result for $\beta = 2$ in Ref. [14] for $g \geq 10$.

The distribution of total transmission has been measured previously by de Boer et al. in optical measurements in slabs of titania particles [13]. Samples with $g > 10^3$ were studied and the distribution was found to be Gaussian to within 1%. A measure of the deviations of the distribution from a Gaussian is the value of the third cumulant $<s_a^3>_c$ which gives the skewness of the distribution and vanishes for a Gaussian distribution. For the samples studied, $<s_a^3>_c$ was of order of 10^{-6}. It was found that $<s_a^3>_c = \gamma_g <s_a^2>_c^2$ with $\gamma_g = 2.9 \pm 0.4$ which is consistent with the value calculated for Gaussian beam excitation of 3.20 [3].

In the present work, low values of g are achieved by placing the sample in a cylindrical copper tube in order to restrict transverse diffusion and thus the number of modes N. The samples consist of randomly positioned polystyrene spheres with diameters of 1.27 cm at a volume filling fraction $f = 0.55$. Transmission spectra were taken at tube diameters of 7.5 and 5.0 cm and various sample lengths in the frequency range 16.8 - 17.8 GHz. The microwave radiation is coupled to the sample by a 0.4 cm wire antenna placed 0.5 cm from the front surface of the sample. The frequency is incremented in 4 MHz steps. The sample tube is rotated between successive measurements to produce new scatterer configurations. The total transmission is measured by use of a single Schottky diode detector positioned inside an integrating sphere which rotates at 2 Hz around the sample axis. The integrating sphere has a diameter of 40 cm and is comprised of two concentric plastic spherical shells separated by a distance of 2 cm. The outer shell is covered with aluminum foilt to form an irregular reflecting surface. The region between the shells is filled with thin-walled aluminum cylinders with diameters of 0.75 cm and typical lengths of 1 cm. The cylinders tumble as the integrating sphere rotates, resulting in fluctuations of the intensity at the detector with a correlation time of ~ 2 ms for the sample with a length of 100 cm. The signal is averaged for 1 s at each frequency, giving an uncertainty of 2.5 % in the measurement of transmission. The signal is normalized by its ensemble average to give s_a. The transmission distributions $P(s_a)$ are obtained by using the data from at least 1000 sample configurations. Distributions obtained using different intervals of the frequency range coincide within experimental error. In the frequency range of the measurements, $\ell \approx 5$ cm and $N = k^2d^2/8 \approx 200$ and 90 for samples in tubes with diameters $d = 7.5$ and 5.0 cm, respectively. The wave number $k = 2\pi/\lambda = 2\pi\nu n/c$ is calculated using an effective medium index of refraction $n \approx 1.4$.

The transmission distributions for three samples with dimensions (a) $d = 7.5$ cm, $L = 66.7$ cm, (b) $d = 5.0$ cm, $L = 50.0$ cm, and (c) $d = 5.0$ cm, $L = 200$ cm are shown in Fig. 1. In the absence of absorption, the dimensionless conductance for these samples without localization corrections, $g = N\ell/L$, would be approximately 15.0, 9.0, and 2.25 for samples a, b, and c, respectively. The distribution broadens and the deviation from a Gaussian becomes more pronounced as either the sample length increases or the cross-sectional area decreases. A value of $<s_a^3>_c$ as large as 0.112 ± 0.003 is observed for sample c. Deviations from a Gaussian distribution in the tail of the distribution for this sample can be seen in the semilog plot of $P(s_a)$ in Fig. 2. For values of $s_a \geq 2$ the distribution is nearly exponential.

In Fig. 3, we present a plot of $<s_a^3>_c$ versus $<s_a^2>_c^2$. The solid line is a least square linear fit to the data which gives $\gamma = 2.38 \pm 0.05$. Within experimental error this equals the value $\gamma = 2.40$ calculated for an incident plane wave in the lowest order of a diagrammatic perturbation expansion in the small parameter $1/g$ [3]. The results are compared to calculations for a plane wave since $d \ll L$ and there is a nearly complete mixing of modes in the sample, giving a uniform average intensity along a cross section of the sample. The agreement between theory and experiment is surprising, however, since $1/g \gtrsim 0.1$ for all samples, reaching a value of approximately 0.3 for sample c, and is by no means small. Furthermore, the influence of absorption was not included in the calculations, whereas the samples used in the experiment are strongly absorbing with $L > L_a \approx 40$ cm, where L_a is the exponential absorption length [15].

We now consider the full transmission distribution. The theoretical expressions for the full distribution function in Refs. [3,4,14] are given as functions of g for nonabsorbing samples. In the present case of strong absorption, the photon number is not conserved, and g cannot be defined in terms of the steady state transmission, while serving as a useful measure of the proximity to the localization transition. This can be seen by noting that the reduction of the average transmission due to absorption would lead to a reduced value of g even though the presence of absorption tends to lessen the degree of correlation in the sample and to push the system farther from the localization threshold. On the other hand, a parameter which characterizes the transmission distribution as well as the closeness to the localization threshold, even in the presence of absorption, is the degree of correlation of intensity in different coherence areas of the transmitted speckle pattern, $<\delta s_{ab}\delta s_{ab'}>$. Were this correlation to vanish, fluctuations in different coherence areas would be independent and the transmission distribution would be Gaussian by the central

limit theorem with $var(s_a) \equiv <s_a^2>_c = 1/N$. As a result of nonlocal correlation, however, the variance of the transmission is enhanced. It is given by $<s_a^2>_c = (<s_{ab}^2>_c - 1)/2 = <\delta s_{ab}\delta s_{ab'}>$ [4,9,10,16]. The last equality is consistent with the results of Ref. [16] when the cumulant intensity correlation function is properly normalized to the renormalized average transmission [17]. In that case, the crossing parameter found by Shnerb and Kaveh [18] which determines the intensity distribution is found experimentally to be equal to $<\delta s_{ab}\delta s_{ab'}>$ [12,16]. The connection of $<\delta s_{ab}\delta s_{ab'}>$ to the full transmission distribution can be seen by considering the expression of Refs. [3,4] for $P(s_a)$ in the absence of absorption in the limit $g \gg 1$,

$$P(s_a) = \int_{-i\infty}^{+i\infty} \frac{dx}{2\pi i} exp[xs_a - \Phi(x)], \tag{1}$$

where

$$\Phi(x) = gln^2(\sqrt{1 + x/g} + \sqrt{x/g})$$

is the generating function. From Eq. (1) one obtains the expression for $<s_a^2>_c$ in terms of g,

$$<s_a^2>_c = \frac{2}{3g}. \tag{2}$$

From these expressions, a general relation for $P(s_a)$ in terms of $<s_a^2>_c$, or equivalently $<\delta s_{ab}\delta s_{ab'}>$, can be found by using Eq. (2) to define a new parameter $g' = 2/3<s_a^2>_c$ which is substituted for g into Eq. (1). Plots of $P(s_a)$ obtained by following this procedure with g' determined from the measured values of $<s_a^2>_c$ are shown as the solid lines in Figs. 1 and 2. We find that $P(s_a)$ is accurately given even for the lowest value of g' of 3.06 (sample c). The distribution of Eq. (1) with g' substituted for g gives the exponential tail, $P(s_a) \sim exp(-g's_a)$ in the limit $s_a \gg 1$. For $s_a \geq 2.0$, the linear fit to the logarithm of the measured transmission distribution for sample c gives a slope of 2.71 ± 0.06 in accord with the exponential fit of the theoretical curve of 2.70 in this range and is quite close to its predicted asymptotic value of 3.06 for $s_a \gg 1$.

The extent of the agreement of Eq. (1) when g' is substituted for g can also be gauged from the comparison between the calculated (circles) and the measured (squares) moments of the transmission distribution shown in Fig. 4 for samples with $g' = 10.2 \pm 0.1$ and $g' = 3.06 \pm 0.04$. The moments calculated from the theory are close to those obtained from the measured distributions. At $n = 10$, these defer by approximately 10 % which is within the experimental error. Thus it appears that $P(s_a)$ can be expressed as a function of the parameter $<s_a^2>_c$.

The agreement between theory and experiment indicates that the ratio of moments is accurately reflected in Eq. (1). The dependence of the second cumulant itself upon sample dimensions is shown in Fig. 5. In the limit $g \gg 1$, in the absence of absorption, $<s_a^2>_c = 2L/3N\ell$. The straight line in the figure is drawn through the first data point and represents $<s_a^2>_c \sim L/N$. As $g \to 1$, and the localization threshold is approached, the scaling theory of localization [19] suggests that g falls more rapidly and hence $<s_a^2>_c$ should increase superlinearly with sample length. Instead, we find that $<s_a^2>_c$ depends sublinearly upon L. This is presumably due to the presence of absorption which diminishes the degree of nonlocal correlation. This raises the question of whether the transmission distribution continues to broaden as L increases or, instead, it reaches a limiting distribution for particular sample parameters.

In conclusion, we have measured the total transmission distribution of microwave radiation in quasi-one-dimensional absorbing samples with small values of g. We find that the distribution can be described using an expression originally derived for nonabsorbing samples in the limit $g \gg 1$ when this expression is reformulated as a function of the single parameter $g' = 2/3 <s_a^2>_c$ determined from the measurements. The validity of the expression for values of g' as small as 3, well beyond the limits assumed in the calculations, may well be associated with the identification of $<s_a^2>_c$ with $<\delta s_{ab}\delta s_{ab'}>$, the degree of spatial correlation in the sample, which is the key microscopic parameter in mesoscopic physics.

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FIGURES:

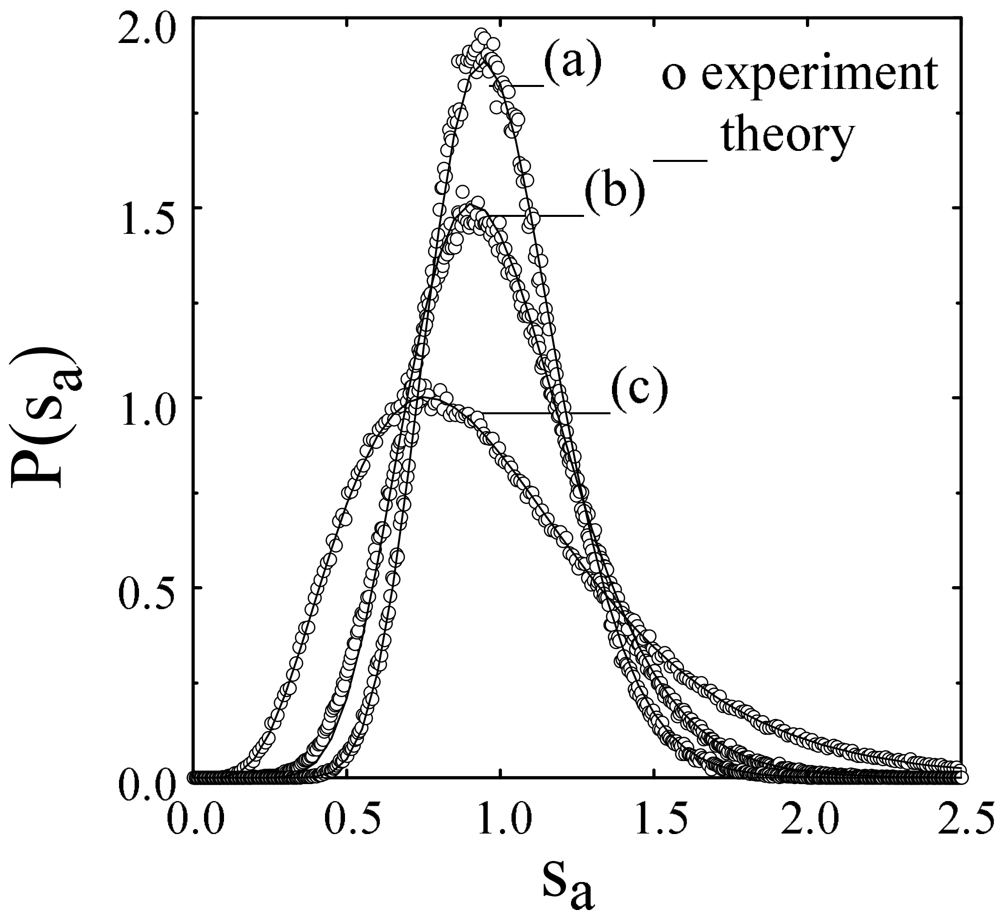
Fig. 1. Distribution function of the normalized transmission $P(s_a)$ for three samples with dimensions (a) $d = 7.5$ cm, $L = 66.7$ cm, (b) $d = 5.0$ cm, $L = 50.0$ cm, and (c) $d = 5.0$ cm, $L = 200$ cm.

Fig. 2. Semilogarithmic plot of the transmission distributions for the same samples as in Fig. 1.

Fig. 3. Plot of $<s_a^3>_c$ versus $<s_a^2>_c^2$. The solid line represents a least square linear fit to the data.

Fig. 4. Comparison of the calculated (circles) and measured (squares) moments of the transmission distribution for samples with (a) $g' = 10.2$ and (b) $g' = 3.06$.

Fig. 5. Dependence of the second cumulant $<s_a^2>_c$ upon sample dimensions.


2.0
1.5
1.0
0.5
0.0
0.0
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1.0
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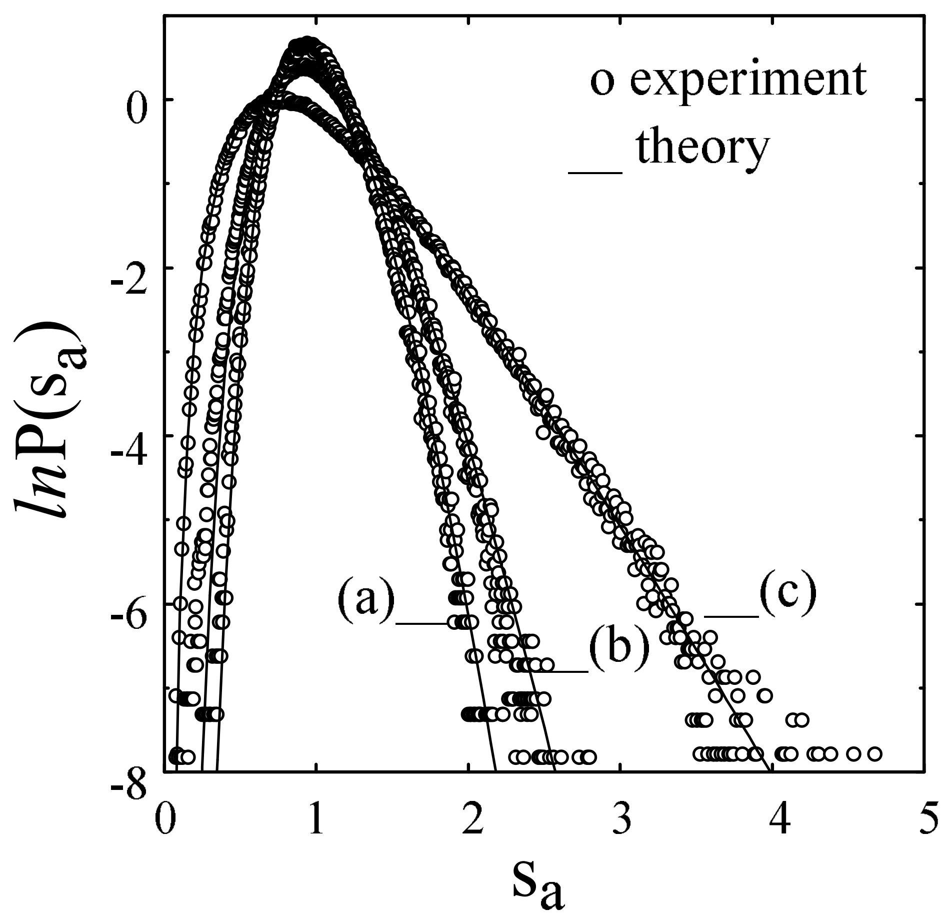
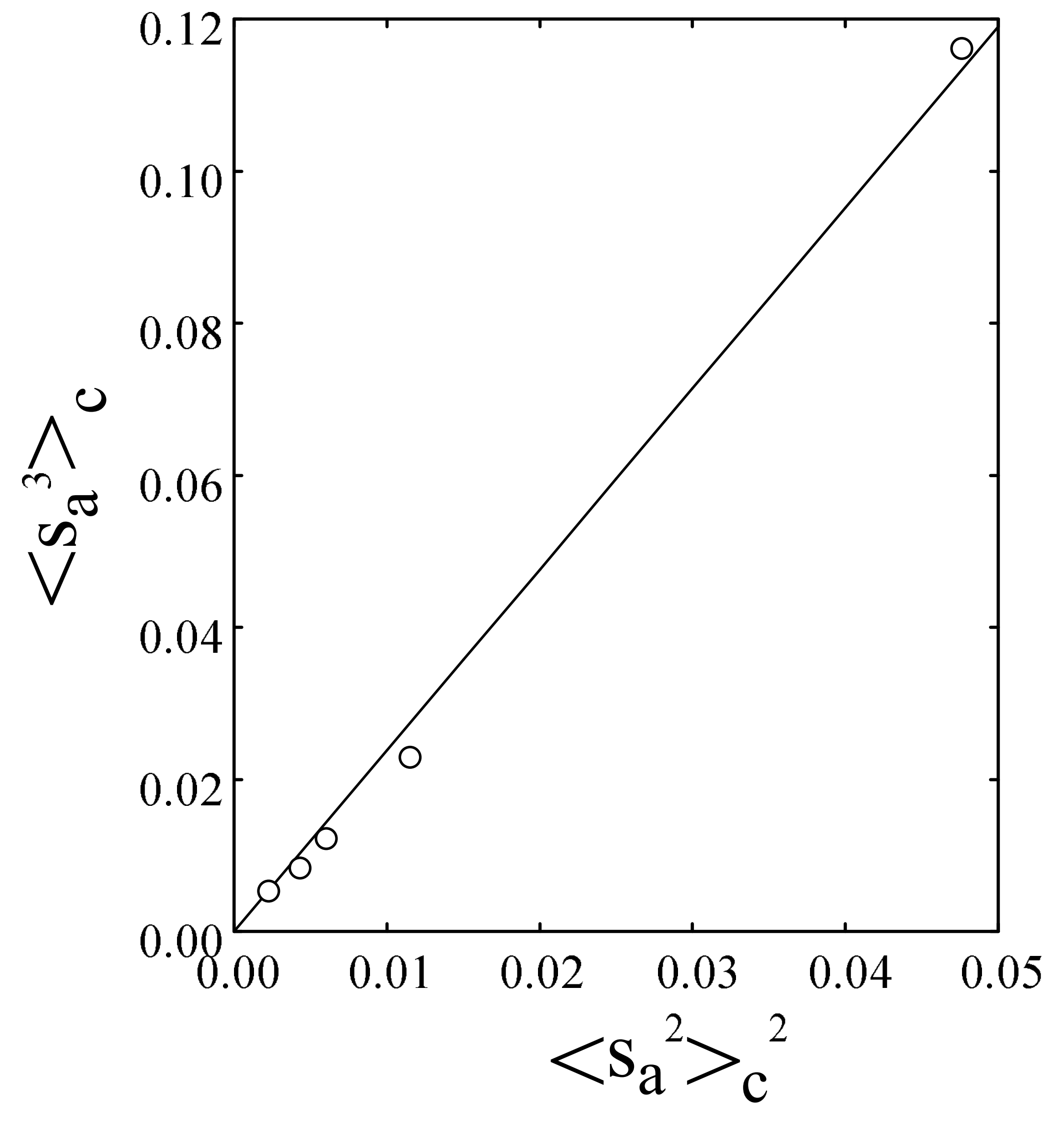
2.5
P(s_a)
s_a
(a)
(b)
(c)
o experiment
___ theory


o experiment
__ theory
(a)
(b)
(c)
lnP(s$_a$)
s$_a$
0
-2
-4
-6
-8
0
1
2
3
4
5


0.12
0.10
0.08
0.06
0.04
0.02
0.00
0.00
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0.02
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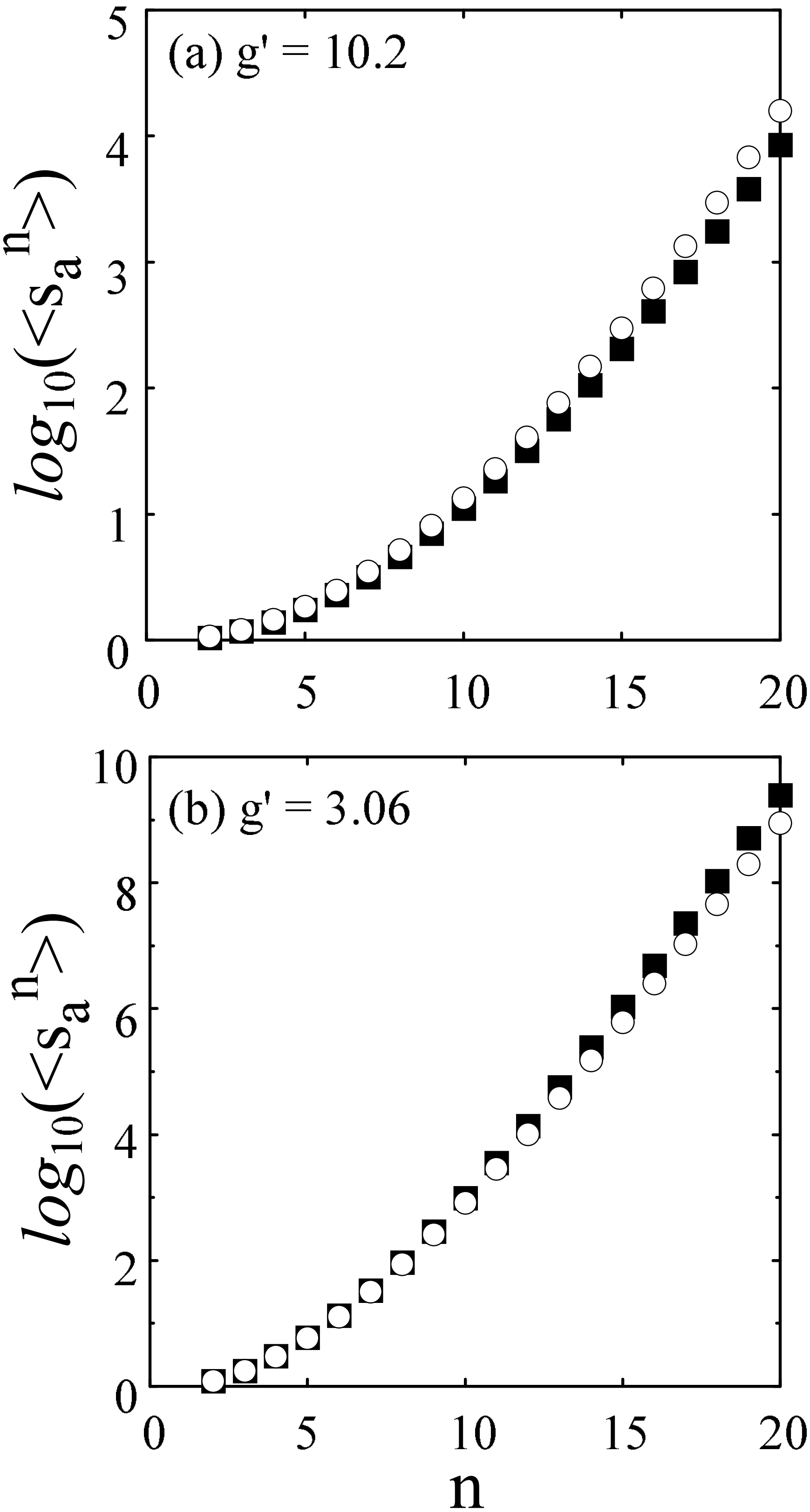
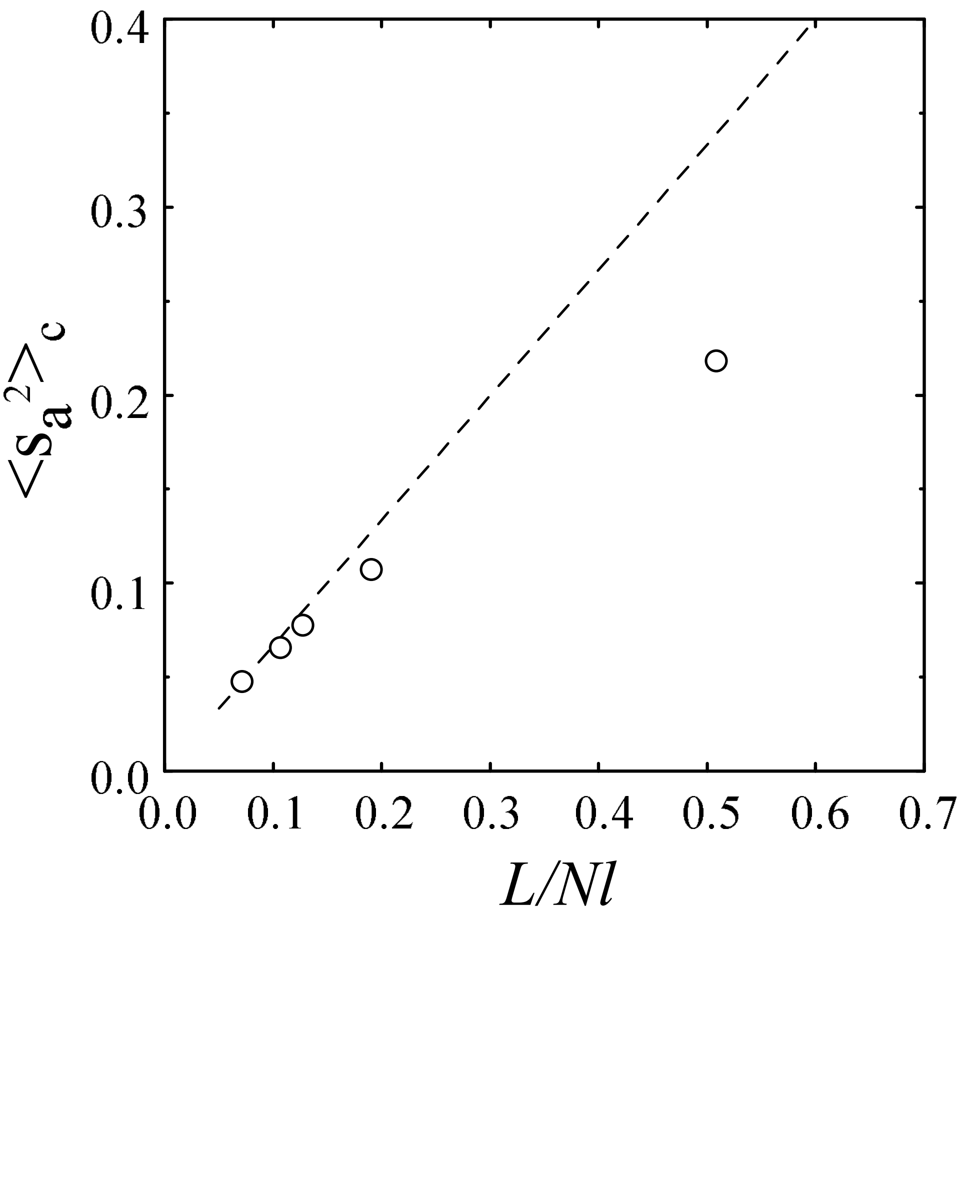
0.05
$\langle s_a^3 \rangle_c$
$\langle s_a^2 \rangle_c^2$



(a) g' = 10.2
(b) g' = 3.06
$log_{10}(<s_a{}^n>)$
n


0.4
0.3
0.2
0.1
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
$<s_a^2>_c$
L/Nl